# Quartic Potential in Phase Space

R. G. G. Amorim[1, *] e W. B. Silva[2, †]

[1]*Universidade de Brasília, Faculdade Gama, 72444-240, Brasília, DF, Brazil.*
[2]*Instituto Federal de Educação, Ciência e Tecnologia de Goiás,*
*Campus de Luziânia 72811-580, Luziânia, GO, Brazil.*

Using the notion of symplectic structure and Weyl product of non-commutative geometry, unitary representations for the Galilei group are construct, and the Schrödinger equation in phase space is derived. An approach for perturbation theory in phase space is derived. The phase space amplitude and the Wigner function are calculated for quartic potential in phase space.

## I. INTRODUCTION

Wigner introduced, in 1932, the first formalism to quantum mechanics in phase space [1]. He was motived by the problem of finding a way to improve the quantum statistical mechanics, based on the density matrix, to treat the transport equations for superfluids. Phase space, $\Gamma$ plays an important role in this realm, since it is the natural manifold to write a kinect theory. In the Wigner formalism, each operator, say $A$, defined in the Hilbert space, $H$, is associated with a function, say $a_w(q,p)$, in $\Gamma$. Then there is an application $\Omega_w : A \to a_w(q,p)$, such that, the associative algebra of operators defined in $H$ turns out to be an associative (but not commutative) algebra is $\Gamma$, given by $\Omega : AB \to a_w(q,p) \star b_w(q,p)$, where the star-product (or Weyl product) $\star$ is defined by Eq.(1) [7]:

$$a_w(q,p) \star b_w(q,p) = a_w(q,p) \exp[\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}]b_w(q,p). \quad (1)$$

In this equation the arrows indicates the direction of action of the operators. Note that Eq.(1) can be seen as an operator $\widehat{A}$ acting on functions $b_w$, such that $\widehat{A}(b_w) = a_w \star b_w$. From a mathematical and physical standpoints, the star-product has been explored in the phase space along different ways[2]-[13]. However, it should be of interest to study the irreducible unitary representations of kinematical groups considering operators of the type $a_w\star$. In this sense, in a recent work [14], using the notion of symplectic struture and Weyl product of a non-commutative geometry, unitary representations of Galilei group were studied and the Schrödinger equation in phase space was obtained. This approach provides a new procedure to derive the Wigner function without the use of the Liouville-von Neumman equation. In an other work [15], this representation was extended to the relativistic case using the Poincaré group. In summary, using the notion of symplectic structure and the Weyl product, unitary representations for Lie algebra for the Poincaré group were constructed. Then the Klein-Gordon and the Dirac equations in phase space are derived. Connection of this formalism with the Wigner function is presented.

In this work, the study is restricted to the non-relativistic case. Eigenvalue problems of Schrödinger equation are studied in phase space using the quartic potential. The problem of harmonic oscillator plus a quartic potential is applicable to numerous physical problems, such as in quantum cosmology [16, 17], quantum chaos [18, 19] and in the theory of crystal [20].

The presentation is organized in the following way. In section 2, we define a Hilbert space $\mathscr{H}(\Gamma)$ over a phase space with its natural symplectic struture. $\mathscr{H}(\Gamma)$ will turn out to be the space of representation of the Galileu group. In the section 3, we construct the generators $a_w(q,p)\star$ for the Galilei group and study the representation space of such operators. In section 4, a representation for the Schrödinger equation in phase space is derived and the lagrangian density is written. In the section 5 results for the quartic oscillator in the pertubative approach are presented. Finally, some closing comments are given in Section 6.

## II. HILBERT SPACE AND SYMPLECTIC STRUTURE

Consider $M$ an $n$-dimensional analytical manifold where each point is specified by coordinates $q = (q^1, ..., q^n)$, such that the coordinates of each point in $T^*M$ will be denoted by $(q,p) = (q^1, ..., q^n, p^1, ..., p^n)$. The space $T^*M$ is equipped with a symplectic struture by introducing a 2-form

$$\omega = dq \wedge dp, \quad (2)$$

called the symplectic form. Consider the following bidifferential operator on $C^\infty(T^*M)$,

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}, \quad (3)$$

such that for $C^\infty$ functions, $f = f(q,p)$ and $g = g(q,p)$, we have

$$\{f,g\} = \omega(f\Lambda, g\Lambda) = f\Lambda g, \quad (4)$$

where

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \quad (5)$$

is the Poisson bracket and $f\Lambda$ and $g\Lambda$ are two vector fields given by

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*ronniamorim@gmail.com
†wassestf@gmail.com

$$X_f = f\Lambda = \frac{\partial f}{\partial q}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}, \tag{6}$$

and

$$X_g = g\Lambda = \frac{\partial g}{\partial q}\frac{\partial}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial}{\partial q}. \tag{7}$$

The space $T^*M$ endowed with this symplectic structure is called the phase space, and will be denoted by $\Gamma$.

The notion of Hilbert space associated with the phase space $\Gamma$ is introduced by considering the set of square integrable functions, $\phi(q,p)$ in $\Gamma$, such that

$$\int dp\,dq\,\phi^*(q,p)\phi(q,p) < \infty. \tag{8}$$

Then we can write $\phi(q,p) = \langle q,p|\phi\rangle$, with

$$\int dp\,dq\,|q,p\rangle\langle q,p| = 1, \tag{9}$$

to be $\langle\phi|$ the dual vector of $|\phi\rangle$. We call this the Hilbert space $\mathscr{H}(\Gamma)$.

## III.   THE GALILEI GROUP IN $\mathscr{H}(\Gamma)$

In this section, the representations of Galilei group in Hilbert space $\mathscr{H}(\Gamma)$ are studied. First we construct unitary transformations $U : H(\Gamma) \to H(\Gamma)$ such that $\langle\psi_1|\psi_2\rangle$ is invariant. Starting with the operator $\Lambda$ defined in Eq.(2) a mapping $e^{\frac{i\hbar\Lambda}{2}} = \star : \Gamma \times \Gamma \to \Gamma$, called Weyl product, can be defined by

$$f(q,p)\star g(q,p) = f(q,p)exp[\frac{i\hbar}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q})]g(q,p), \tag{10}$$

where $f$ and $g$ are in $\Lambda$ and $\partial_x = \frac{\partial}{\partial x}$, $x = (q,p)$ ( The reduced Planck constant is used to fix units.)

The Galilei Lie algebra in phase space can be constructed by using the Weyl operators given by $f\star$. A set of operators are defined without implying a physics interpretation. We define the following operators,

$$\widehat{Q} = q\star = q + \frac{i\hbar}{2}\partial_p, \tag{11}$$

and

$$\widehat{P} = p\star = p - \frac{i\hbar}{2}\partial_q. \tag{12}$$

Using the functions $k_i$,

$$k_i = mq_i - tp_i, \tag{13}$$

where $m$ and $t$ are parameters, the star-operator relative this function is

$$\widehat{K} = k_i\star = mq_i\star - tp_i\star = m\widehat{Q}_i - t\widehat{P}_i. \tag{14}$$

For the functions

$$l_i = \varepsilon_{ijk}q_jp_k, \tag{15}$$

we have the star-operator,

$$\begin{aligned}\widehat{L}_i = \varepsilon_{ijk}\widehat{Q}_j\widehat{P}_k &= \varepsilon_{ijk}q_jp_k - \frac{i\hbar}{2}\varepsilon_{ijk}q_j\frac{\partial}{\partial p_k}\\ &+ \frac{i\hbar}{2}\varepsilon_{ijk}p_k\frac{\partial}{\partial q_j} + \frac{\hbar^2}{4}\frac{\partial^2}{\partial q_j\partial p_k}\end{aligned} \tag{16}$$

For the function

$$h = \frac{\mathbf{p}^2}{2m} = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2), \tag{17}$$

the star-operator is given as

$$\begin{aligned}\widehat{H} &= \frac{\widehat{P}^2}{2m} = \frac{1}{2m}(\widehat{P}_1^{\,2} + \widehat{P}_2^{\,2} + \widehat{P}_3^{\,2})\\ &= \frac{1}{2m}[(p_1 - \frac{i\hbar}{2}\frac{\partial}{\partial q_1})^2 + (p_2 - \frac{i\hbar}{2}\frac{\partial}{\partial q_2})^2\\ &\quad + (p_3 - \frac{i\hbar}{2}\frac{\partial}{\partial q_3})^2]\end{aligned} \tag{18,19}$$

These operators satisfy the Galilei-Lie algebra with central extension. The Galilei-Lie algebra is given for the commutation relations,

$$[\widehat{L}_i, \widehat{L}_j] = i\hbar\varepsilon_{ijk}\widehat{L}_k, \tag{20}$$

$$[\widehat{L}_i, \widehat{K}_j] = i\hbar\varepsilon_{ijk}\widehat{K}_k, \tag{21}$$

$$[\widehat{L}_i, \widehat{P}_j] = i\hbar\varepsilon_{ijk}\widehat{P}_k, \tag{22}$$

$$[\widehat{K}_i, \widehat{K}_j] = 0, \tag{23}$$

$$[\widehat{K}_i, \widehat{P}_j] = i\hbar m\delta_{ij}\mathbf{1}, \tag{24}$$

$$[\widehat{K}_i, \widehat{H}] = i\hbar\widehat{P}_i, \tag{25}$$

$$[\widehat{P}_i, \widehat{P}_j] = 0, \tag{26}$$

$$[\widehat{P}_i, \widehat{H}] = 0, \tag{27}$$

$$[\widehat{L}_i, \widehat{H}] = 0. \tag{28}$$

For the Galilean symmetry, $\widehat{P}$, $\widehat{K}$, $\widehat{L}$ and $\widehat{H}$ are generators of translations, boost, rotations and time translations. The physical content of this representation is derived by observing that $\widehat{Q}$ and $\widehat{P}$ are transformed by the boost according to:

$$exp(-i\mathbf{v}.\frac{\widehat{K}}{\hbar})\widehat{P}_j exp(i\mathbf{v}.\frac{\widehat{K}}{\hbar}) = \widehat{P}_j + mv_j\mathbf{1}, \tag{29}$$

$$exp(-i\mathbf{v}.\frac{\widehat{K}}{\hbar})\widehat{Q}_j exp(i\mathbf{v}.\frac{\widehat{K}}{\hbar}) = \widehat{Q}_j + v_jt\mathbf{1}. \tag{30}$$

Futhermore

$$[\widehat{Q}_j, \widehat{P}_n] = i\hbar\delta_{jn}\mathbf{1}, \tag{31}$$

Therefore, $\widehat{Q}$ and $\widehat{P}$ can be taken to be the physical observables of position and momentum, respectively, with Eq.(29) and Eq.(31) describing, consistently, the way $\widehat{Q}$ and $\widehat{P}$ transform under the Galilei boost.

The invariants of the Galilei algebra in this representation are given by

$$I_1 = \widehat{H} - \frac{\widehat{P}^2}{2m} \quad and \quad I_2 = \widehat{L} - \frac{1}{m}\widehat{K} \times \widehat{P}. \tag{32}$$

The invariant $I_1$ describes the Hamiltonian of free particle, while $I_2$ is associated with the spin degrees of freedom. The parameters $m$ and $t$ are interpreted as mass and time. Here, we will be mainly concerned with the scalar representations i.e. $I_2 = 0$.

With $\widehat{H}$, the time evolution of an observable $\widehat{A}$ is specified by

$$exp(-it\frac{\widehat{H}}{\hbar})\widehat{A}(0)exp(it\frac{\widehat{H}}{\hbar}) = \widehat{A}(t), \tag{33}$$

which results in

$$i\hbar\frac{\partial \widehat{A}(t)}{\partial t} = \widehat{A}(t)\widehat{H} - \widehat{H}\widehat{A}(t) = [\widehat{A}(t), \widehat{H}]. \tag{34}$$

The operators of position and momentum are defined by the following form,

$$\widehat{P} = p\star = p\mathbf{1} - \frac{i\hbar}{2}\partial_q = p\mathbf{1} + \frac{1}{2}\widetilde{P}, \tag{35}$$

and

$$\widehat{Q} = q\star = p\mathbf{1} + \frac{i\hbar}{2}\partial_p = q\mathbf{1} + \frac{1}{2}\widetilde{Q}. \tag{36}$$

If the c-*number* operators are defined as

$$\overline{P} = 2p\mathbf{1} \quad e \quad \overline{Q} = 2q\mathbf{1}, \tag{37}$$

the position and momentum operators can be written as

$$\widehat{P} = \frac{1}{2}(\overline{P} + \widetilde{P}) \quad and \quad \widehat{Q} = \frac{1}{2}(\overline{Q} + \widetilde{Q}). \tag{38}$$

In accord with the boost, $\overline{Q}$ and $\overline{P}$ transform as

$$exp(-iv\frac{\widehat{K}}{\hbar})2\overline{Q}exp(iv\frac{\widehat{K}}{\hbar}) = 2\overline{Q} + vt\mathbf{1}, \tag{39}$$

and

$$.exp(-iv\frac{\widehat{K}}{\hbar})2\overline{P}exp(iv\frac{\widehat{K}}{\hbar}) = 2\overline{P} + mv\mathbf{1}. \tag{40}$$

Consequently, we find that $\overline{Q}$ and $\overline{P}$ transform as position and momentum. However, since $[\overline{Q}, \overline{P}] = 0$, $\overline{Q}$ and $\overline{P}$ cannot be interpreted as observables, althought they can be used to construct a phase space frame in the Hilbert space. So, a set of normalized eigenvectors, $|q, p\rangle$, are defined with $\{q\}$ and $\{p\}$ ,being a set of eigenvalues, that satisfy

$$\overline{Q}|q, p\rangle = q|q, p\rangle, \tag{41}$$

and

$$\overline{P}|q, p\rangle = p|q, p\rangle, \tag{42}$$

with

$$\langle q, p|q', p'\rangle = \delta(q - q')\delta(p - p'), \tag{43}$$

and

$$\int dqdp|q, p\rangle\langle q, p| = 1. \tag{44}$$

The operators $\overline{Q}$ and $\overline{P}$, with the eigenvalues $\{q, p\}$ are coordinates of a phase space $\Gamma$, where the symplectic struture is used to define the Weyl product. Using the operator $\Lambda$, the star product is constructed as

$$e^{\frac{i\hbar}{2}\Lambda} : \Gamma \times \Gamma \to \Gamma. \tag{45}$$

Thus the representations of Galilei group, that were constructed,provide a structure in the simplectic manifold.

## IV. THE SCHRÖDINGER EQUATION IN PHASE SPACE

Consider $|\alpha(t)\rangle$ in $H(\Gamma)$ as a representative quantity describing the state of a quantum system, such that with the kets $\{|q, p\rangle\}$, we have

$$\psi_\alpha(q, p, t) = \langle q, p|\alpha, t\rangle. \tag{46}$$

It is important to note that $\psi_\alpha(q, p, t)$ is a wave function but not with the content of the usual quantum mechanics state, when $q$ and $p$ are the eigenvalues of the ancillary operators $\overline{Q}$ and $\overline{P}$.

The time evolution of the wave function is given by

$$\psi(q, p, t) = \exp\frac{-itH}{\hbar} \star \psi(q, p, 0), \tag{47}$$

From these relations the following equation is derived(using the usual form for the Hamiltonian, $H = h + V(q) = \frac{p^2}{2m} + V(q)$),

$$i\hbar\partial_t\psi = (\frac{p^2}{2m} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial q^2} - \frac{i\hbar p}{2m}\frac{\partial}{\partial q})\psi + V(q + \frac{i\hbar}{2}\frac{\partial}{\partial p})\psi; \tag{48}$$

which is the Schrödinger equation in phase space.

The lagrangian that leads to equation above is given by

$$\mathscr{L} = \frac{i\hbar}{2}(\psi^\dagger\partial_t\psi - \psi\partial_t\psi^\dagger) + \frac{i\hbar}{4m}p(\psi^\dagger\partial_q\psi - \psi\partial_q\psi^\dagger)$$
$$- \frac{p^2}{2m}\psi\psi^\dagger + V(q) \star (\psi\psi^\dagger) - \frac{\hbar^2}{8m}\partial_q\psi\partial_q\psi^\dagger. \tag{49}$$

The association with the Wigner function is given by

$$f_w(q,p) = \psi(q,p,t) \star \psi^{\dagger}(q,p,t). \tag{50}$$

The wave functions, $\psi(q,p)$, obeys the follow eigenvalues equation,

$$H \star \psi(q,p) = E\psi(q,p), \tag{51}$$

where $H$ is the hamiltonian.

This equation is similar to the eigenvalue equation satisfied by Wigner equation, and we found that $f_w(q,p)$ and $\psi(q,p)$ satisfy the same differential equation.

## V.   QUARTIC POTENTIAL

### A.   The quartic hamiltonian

The hamiltonian of the harmonic oscillator potential is written by

$$\widehat{H} = \frac{1}{2m}\widehat{P}^2 + m\omega^2\widehat{Q}^2,$$

where the position and momentum operators are given by Eq.(11) and Eq.(12) respectively. The hamiltonian for the anharmonic oscillator is given by

$$\widehat{H} = \frac{1}{2m}\widehat{P}^2 + m\omega^2\widehat{Q}^2 + \alpha\widehat{Q}^4. \tag{52}$$

If $\widehat{Q}$ and $\widehat{P}$ are written in terms of annihilation and creation operators [14], we have

$$\widehat{A} = \sqrt{\frac{m\omega}{2\hbar}}(\widehat{Q} + \frac{i}{m\omega}\widehat{P}), \tag{53}$$

$$\widehat{A}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(\widehat{Q} - \frac{i}{m\omega}\widehat{P}), \tag{54}$$

and the corresponding hamiltonian is

$$\widehat{H} = \hbar\omega(\widehat{A}\widehat{A}^{\dagger} - \frac{1}{2}) + \frac{\hbar^2}{4m^2\omega^2}(\widehat{A} + \widehat{A}^{\dagger})^4. \tag{55}$$

The operators $\widehat{A}$ and $\widehat{A}^{\dagger}$ satisfy the following commutation relation,

$$[\widehat{A}, \widehat{A}^{\dagger}] = 1, \tag{56}$$

and define $\widehat{H}_0 = \hbar\omega(\widehat{A}\widehat{A}^{\dagger} - \frac{1}{2})$.

The creation and annihilation operators are applied to the eigenfunctions of the harmonic oscillator in phase space with the unperturbated wave functions $\psi^0(q,p)$, lead to the following relations,

$$\widehat{A}\psi_n^{(0)}(q,p) = \sqrt{n}\psi_{n-1}^{(0)}(q,p), \tag{57}$$

and

$$\widehat{A}^{\dagger}\psi_n^{(0)}(q,p) = \sqrt{n+1}\psi_{n+1}^{(0)}(q,p). \tag{58}$$

The creation and annihilation operators may be written as [14]

$$\widehat{A} = \sqrt{\frac{m\omega}{2\hbar}}[(q + \frac{i\hbar}{2}\overrightarrow{\partial}_p) + \frac{i}{m\omega}(p - \frac{i\hbar}{2}\overrightarrow{\partial}_q)], \tag{59}$$

and

$$\widehat{A}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}[(q + \frac{i\hbar}{2}\overrightarrow{\partial}_p) - \frac{i}{m\omega}(p - \frac{i\hbar}{2}\overrightarrow{\partial}_q)]. \tag{60}$$

### B.   Perturbative approach

The Schrödinger equation in phase space with quartic potential will be considered in a perturbative approach to first order. The state function with zero upper index represents the state function of unperturbed harmonic oscillator.

The unperturbed solutions of the Schrödinger equation in phase space is given by,

$$\widehat{H}_0\psi_n^{(0)}(q,p) = E_n^{(0)}\psi_n^{(0)}(q,p). \tag{61}$$

Now we introduced the perturbation $V(\widehat{Q}) = \lambda\widehat{Q}^4$, where $\lambda$ is a small parameter. If $\psi_n(q,p)$ is the state function of perturbated system, we have,

$$\widehat{H}\psi_n(q,p) = (\widehat{H}_0 + \lambda V)\psi_n(p,q) = E_n\psi_n(q,p). \tag{62}$$

However, if the state functions $\psi_n^{(0)}$ are known, the approximate value for $\psi_n$ and $E_n$ can be calculate by the perturbative approach. To first order, suppose that $\psi_n$ and $E_n$ can written as,

$$\psi_n = \psi_n^{(0)} + \lambda\psi_n^{(1)}, \tag{63}$$

and

$$E_n = E_n^{(0)} + \lambda E_n^{(1)}, \tag{64}$$

where $\psi_n^{(1)}$ and $E_n$ is the first order correction for the state function and the energy respectively.

Thus ignoring the quadratic term in $\lambda$, we obtain

$$(\widehat{H}_n^{(0)} - E_n^{(0)})\psi_n^{(1)} = (E_n^{(1)} - V)\psi_n^{(0)}. \tag{65}$$

Now expanding $\psi_n^{(1)}$ in terms of the state function of the unperturbed system we have

$$\psi_n^{(1)} = \sum_k a_k\psi_k^{(0)}. \tag{66}$$

Combining these equations, we obtain

$$\sum_k a_k(E_k^{(0)} - E_n^{(0)})\int \psi_m^{(0)\dagger} \star \psi_k^{(0)}dqdp = \int \psi_m^{(0)\dagger}(E_n^{(1)} - V)\psi_n^{(0)}dqdp$$
$$\tag{67}$$

However the state functions of the unperturbed system are orthogonal, i.e.

$$\int \psi_m^{(0)\dagger} \star \psi_n^{(0)} dq dp = \delta_{mn}, \tag{68}$$

so the Eq.(67) can be written by

$$a_m(E_m^{(0)} - E_n^{(0)}) = E_n^{(1)} \delta_{mn} - \int \psi_m^{(0)\dagger} V \psi_n^{(0)} dq dp. \tag{69}$$

Now, we consider two cases, when $m = n$ and when $m \neq n$. (i) For $m = n$ the Eq.(69) give us the first order correction to the energy, given by

$$E_n^{(1)} = \int \psi_n^{(0)\dagger} V \psi_n^{(0)} dq dp = \langle V \rangle. \tag{70}$$

For $m \neq n$, the Eq.(69) give us,

$$a_m = \frac{\int \psi_m^{(0)\dagger} V \psi_n^{(0)} dq dp}{(E_n^{(0)} - E_m^{(0)})}. \tag{71}$$

Using Eq.(66), we find that

$$\psi_n^{(1)} = \sum_{m \neq n} \frac{\int \psi_m^{(0)\dagger} V \psi_n^{(0)} dq dp}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}. \tag{72}$$

Finally, in the first approximation, the state function $\psi_n$ for perturbed system is written as

$$\psi_n(q,p) = \psi_n^{(0)}(q,p) + \sum_{m \neq n} \frac{\int \psi_m^{(0)\dagger}(q,p) V \psi_n^{(0)}(q,p) dq dp}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}(q,p) \tag{73}$$

### C. Amplitudes in phase space and Wigner function

In order to calculate the state function of the quartic potential in phase space. Using the creation and annihilation operators, Eq.(57) and Eq.(58), with the relations given in Eq.(71) and Eq.(73), we obtain

$$\begin{aligned}
\psi_n^{(1)}(q,p) &= \frac{1}{8}\left[ \frac{\sqrt{n(n-1)(n-2)(n-3)}}{2} \psi_{n-4}^{(0)} + \left(\sqrt{n(n-1)^3} \right.\right. \\
&+ \left.\sqrt{n(n-1)(n-2)^2} + \sqrt{n^3(n-1)}\right) \psi_{n-2}^{(0)} \\
&- \left(\sqrt{n^2(n+1)(n+2)} + \sqrt{(n+2)(n+1)^3}\right. \\
&+ \left.\sqrt{(n+1)(n+2)(n+3)^2} + \sqrt{n(n+2)^3}\right) \psi_{n+2}^{(0)} \\
&- \left.\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{2} \psi_{n+4}^{(0)} \right]. \tag{74}
\end{aligned}$$

And the amplitude in phase space for the quartic potential is written in following form,

$$\begin{aligned}
\psi_n(q,p) &= \psi_n^{(0)} + \frac{1}{8}\left[ \frac{\sqrt{n(n-1)(n-2)(n-3)}}{2} \psi_{n-4}^{(0)} \right. \\
&+ \left(\sqrt{n(n-1)^3} + \sqrt{n(n-1)(n-2)^2} + \sqrt{n^3(n-1)}\right) \psi_{n-2}^{(0)} \\
&- \left(\sqrt{n^2(n+1)(n+2)} + \sqrt{(n+2)(n+1)^3}\right. \\
&+ \left.\sqrt{(n+1)(n+2)(n+3)^2} + \sqrt{n(n+2)^3}\right) \psi_{n+2}^{(0)} \\
&- \left.\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{2} \psi_{n+4}^{(0)} \right], \tag{75}
\end{aligned}$$

where the state function for the unperturbed harmonic oscillator are well known [14].

The correspondent Wigner function are calculate by the following expression

$$f_w(q,p) = \psi_n(q,p) \star \psi_n^{\dagger}(q,p). \tag{76}$$
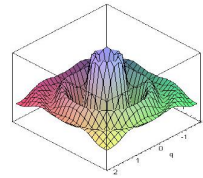


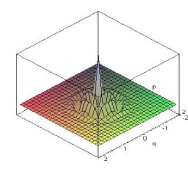FIG. 1: Amplitude for quartic potential, n=4



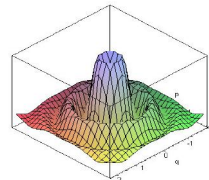FIG. 2: Wigner function for quartic potential, n=4



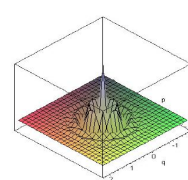FIG. 3: Amplitude for quartic potential, n=5



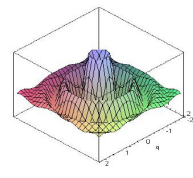FIG. 4: Wigner function for quartic potential, n=5



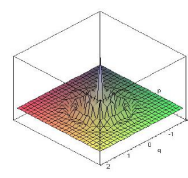FIG. 5: Amplitude for quartic potential, n=6



FIG. 6: Wigner function for quartic potential, n=6

The behavior of the phase space amplitudes and the respective Wigner functions are shown in figures (1)- (6).

## VI.  CONCLUDING REMARKS

A brief review of derivation of the Schrödinger equation in phase space from the representations of the Galilei group, using the star product. The analysis of the quartic potential is made using a perturbative approach to first order. The central point is to obtain the Wigner function without use the Liouville-von Neumman equation. This suggest that application to physical problems in cosmology, gravity and chaos can be considered in detail. Future developments may concern the development of the scattering theory in phase space.

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