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Geometry and Physics

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Abstract

In this work, the student learned to reformulate classical results of mechanics in terms of differential geometry, the mathematical framework of modern physics. This will allow the undergraduate student learn the tool to tackle higher level physics not covered in undergraduate courses.

Keywords:

Smooth manifolds, Classical mechanics, Symplectic geometry

Introduction

Differential geometry is used in many areas of physics, such as general relativity, gauge theories, and even classical mechanics. In this work we use this to study the symplectic formulation of classical mechanics.

Results and Discussion

Physical models are often formulated in terms of differential geometry. The idea of curved spaces is generalized by locally euclidean spaces, called manifolds:

Definition: ***M*** *is said to be a* <u>*manifold*</u> *if it is a Hausdorff, second countable, topological space and for all* ***pϵM*** *there's a neighborhood* ***U*** *with a homeomorphism* $f_U : U \to R^n$*. The colection of all such neighborhoods and homeomorphisms is called an atlas.*
M *is a* <u>*smooth manifold*</u> *if its atlas is* C^∞ *compatible.*

Vector and co-vector fields are generalized by smooth sections of the tangent and cotangent bundles, respectively:

Definition: *The* <u>*tangent bundle*</u> ***TM*** *of* ***M*** *is the set of derivations of local observable. For each* ***pϵM****, the tangent space* T_pM *of tangent vectors over* ***p*** *is a vector space. By considering each dual space* T_pM^**, the* <u>*cotangent bundle*</u> ***T*M*** *is the disjoint union of each* T_pM^**.*

We also have the idea of the flow of a vector field:

Theorem: *Let* ***X*** *be a smooth vector field over* ***M****. The differential equation* $\partial\varphi(t,y)/\partial t = X_{\varphi(t,y)}$ *admits a local solution for every* ***yϵM****, called the* <u>*flow*</u> *of* ***X****.*

In Lagrangian mechanics, the trajectory of particles are the extremes of the action functional $\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$ and so satisfy the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

The Hamiltonian formalism is derived by the following theorem:

Theorem: *The Euler-Lagrange equation are equivalent to Hamilton equations:*

$\dot{q}^i = \frac{\partial H}{\partial p_i}$ *and* $\dot{p}_i = -\frac{\partial H}{\partial q^i}$ *where* $p = \frac{\partial L}{\partial \dot{q}}$ *and* $H(p,q,t) = p\dot{q} - L(q,\dot{q},t)$ *is the Legendre transform of the Lagrangian.*

However, by considering the cotangent bundle **T*M** as the system's phase space and the symplectic Liouville 2-form $\sum_{i=0}^{n} \omega = dq^i \wedge dp_i$, we have a natural isomorphism between co-vector and vector fields over **T*M**, the Hamilton equations appear naturally:

Definition: *The* <u>*Hamiltonian field*</u> X_H *of an observable* ***H*** *is the vector field such that* $dH(X) = \omega(X_H, X)$ *for all vector fields* ***X****.*

Theorem: *The flow of the Hamiltonian field of any time-independent observable* ***H*** *satisfies the Hamilton equations.*

In this context, the Hamiltonian is always a conserved quantity. With the Poisson bracket, we can calculate how any observable of the phase space evolves in this motion by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}\right).$$

We see that the symplectic structure of the cotangent bundle is a very natural framework for studying classical mechanics.

Conclusions

By using the language of differential geometry, we were able to reformulate Hamiltonian mechanics very naturally on the symplectic structure of the phase space.
Equiped with this technique, the student will follow up by studying general relativity and gauge theories, for which differential geometry is an essential tool.

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[1] Rivasseau, V. **Géométrie et Physique.** Ecole Polytechnique; 1993.